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Soliton Equations as Dynamical Systems on a Infinite Dimensional Grassmann Manifolds.

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§0. Introduction.

It is shown that the totality of solutions to Kadomtsev-Petviashvili (KP) equation

$$3u_{yy} + (-4u_t+u_{xxx}+6uu_x)_x = 0 \qquad (0.1)$$

has a natural structure of Grassmann manifold (GM) of infinite dimension. Evolution of u in variables x,y,t (and also hidden "higher variables") is now interpreted as dynamical motion of a point on the GM by the action of 3 (or more) parameter subgroup of the group $GL(\infty)$ of automorphisms of our GM. Generic points of this GM give generic solutions to KP equation, whereas points on particular submanifold of GM give solutions of particular type — such as rational solutions, (multi-)soliton solutions, (multi-phase) quasi-periodic solutions, or (multi-phase) similarity solutions. (E.g. rational solutions correspond to points on finite dimensional Grassmann manifold contained in our infinite GM.) Also, different kinds of submanifolds of the same GM (and in some cases different choice of parameter subgroup of $GL(\infty)$) give rise to generic solutions of other soliton equations such as Korteweg - de Vries (KdV) equation, modified KdV equation, Boussinesq equation, Sawada - Kotera equation, Non-linear Schrödinger equation, Toda lattice, equation of self induced transparency, Benjamin - Ono equation, etc. as well as to solutions of particular type of these soliton equations.

The automorphism group $GL(\infty)$ of GM naturally plays the role of group of transformations (or, of "hidden symmetries") of KP equation. This fact provides us as a consequence with thourough understanding on transformation property of various soliton equations.

Multi-component generalization of the theory shows that other soliton equations such as equation of three wave interaction, multi-component non-linear Schrödinger equation, Sine-Gordon equation, Lund - Regge equation, equation for intermediate long wave, etc. also constitute submanifolds of infinite dimensional GM.

It may be conjectured that <u>any</u> soliton equation, or completely integrable system, is obtained in this way. Classification of soliton equations would then be reduced to classification of submanifolds of our GM which are stable by the subgroup of $GL(\infty)$ describing space-time evolution.

§1. KP equation as isospectral deformation equation of a micro-differential operator.

Consider a formal micro-differential (or pseudo-differential) operator

$$L \underset{\text{def}}{=} \sum_{n=0}^{\infty} u_n(x)\left(\frac{d}{dx}\right)^{1-n}, \qquad \text{with } u_0(x) = 1,$$

$$= \frac{d}{dx} + u_1(x) + u_2(x)\left(\frac{d}{dx}\right)^{-1} + \dots \quad . \tag{1.1}$$

Since $u_1(x)$ is easily eliminated by the simple transformation $L \mapsto e^{S(x)} L\, e^{-S(x)}$ with $S(x) \underset{\text{def}}{=} \int^x u_1(x)dx$, we hereafter set $u_1(x) = 0$ without loss of generality. Let B_n denote the "differential operator" part of the n-th power L^n of L, e.g. $B_0 = 1$, $B_1 = \frac{d}{dx}$, $B_2 = (\frac{d}{dx})^2 + 2u_2$, $B_3 = (\frac{d}{dx})^3 + 3u_2(\frac{d}{dx}) + (3u_3 + 3u_{2,x}), \dots$, then isospectral deformation of the eigenvalue problem

$$L\psi = \lambda\psi, \qquad \lambda = \text{eigenvalue}, \tag{1.2}$$

is achieved by

$$\frac{\partial}{\partial t_n}\psi = B_n\psi, \quad n = 0,1,2,\dots, \tag{1.3}$$

where t_0, t_1, $t_2, \dots$ are deformation parameters on which the quantities $u_2(x)$, $u_3(x), \dots$ and $\psi(x,\lambda)$ now depend, or equivalently, isospectral deformation of L is given by Lax-type equations

$$\frac{\partial L}{\partial t_n} = [B_n, L] \ ; \qquad n = 0,1,2,\ldots \tag{1.4}$$

(1.4) are equivalent to Zakharov-Shabat equations which derive from (1.3):

$$[B_n - \frac{\partial}{\partial t_n}\ ,\ B_m - \frac{\partial}{\partial t_m}] = 0; \qquad n,m = 0,1,2,\ldots\ . \tag{1.5}$$

The deformation equations ((1.4) or (1.5)) imply in particular that L and hence its coefficients $u_n(x;t)$ do not depend on t_0 and depend on t_1 and x via their sum $x+t_1$, i.e.

$$u_n = u_n(x+t_1,\ t_2,\ t_3,\ldots), \tag{1.6}$$

and hence, by replacing $x+t_1$ by t_1, we consistently identify the variable x with t_1 and $\frac{d}{dx}$ with $\frac{\partial}{\partial t_1}$.

The deformation equations now read

$$\begin{aligned} &u_{2,t_2} = 2u_{3,x}+u_{2,xx},\ u_{3,t_2} = 2u_{4,x}+u_{3,xx}+2u_2u_{2,x} \\ &u_{2,t_3} = 3u_{4,x}+3u_{3,xx}+u_{2,xxx}+6u_2u_{2,x},\ldots \end{aligned} \tag{1.7}$$

which yield in particular the KP equation (0.1) for u_2:

$$3u_{2,t_2t_2} = (-4u_{2,t_3}+u_{2,xxx}+6u_2u_{2,x})_x = 0$$

where x,y,t are now written $x(=t_1)$, t_2, t_3. The complete set of equations (1.7) describes complete hierarchy of extended or higher KP equations involving the complete set of deformation parameters $t_1, t_2, t_3, t_4, \dots$.

Within the category of microdifferential operators, our L of (1.1) is transformed to the trivial one, $\frac{d}{dx}$. Namely we can find

$$W = 1 + w_1(x)(\frac{d}{dx})^{-1} + w_2(x)(\frac{d}{dx})^{-2} + \dots \qquad (1.8)$$

so that

$$L = W\cdot\frac{d}{dx}\cdot W^{-1} \qquad (1.9)$$

holds. Accordingly the eigenfunction ψ is also transformed to the trivial one, ψ_0 = constant $\times\ e^{\lambda x+(t_0+\ t_1+\ {}^2t_2+\dots)}$, yielding

$$(x,t) = W\psi_0(x,t)$$

$$= (1+w_1(x,t)\lambda^{-1}+w_2(x,t)\lambda^{-2}+\dots)e^{\lambda x+(t_0+\lambda t_1+\lambda^2t_2+\dots)} \qquad (1.10)$$

except for an arbitrary constant factor.

§2. Generalization to multi-component case

The construction of KP hierarchy given above is generalized to multi-component case as follows. (The number of components will be denoted by r. When $r=1$ the following formalism reduces to the previous one.)

We introduce formal microdifferential operators with matrix-coefficients of size $r \times r$:

$$L = \sum_{n=0}^{\infty} U_n(x)\left(\frac{d}{dx}\right)^{1-n}, \quad \text{with } U_0(x) = 1, \quad U_1(x) = 0, \tag{2.1}$$

$$C_i = \sum_{n=0}^{\infty} C_{in}(x)\left(\frac{d}{dx}\right)^{-n}, \quad \text{with } C_{i0}(x) = E_{ii} = \begin{array}{c} \quad\ i \\ \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \end{array}, \tag{2.2}$$

$$(i=1,\ldots,r)$$

subject to the conditions (i) L commutes C_i, and (ii) $C_i C_j = \delta_{ij} C_i$. Entries of $U_n(x)$ and $C_{in}(x)$ will play the role of "potentials", i.e. unknown functions to be solved in our multi-component version of KP hierarchy.

Consider the following simultaneous eigenvalue problem of $r \times r$ matrix Ψ

$$\begin{cases} L\Psi = \lambda\Psi, & \lambda = \text{eigenvalue}, \quad (2.3) \\ C_i\Psi = \Psi E_{ii}, & (i = 1,\ldots r) \quad (2.4) \end{cases}$$

i.e. the j-th column vector $\Psi^{(j)}$ of Ψ is a simultaneous eigenvector of $L, C_1, \ldots, C_r$:

$$L\Psi^{(j)} = \lambda\Psi^{(j)}, \quad C_i\Psi^{(j)} = \delta_{ij}\Psi^{(j)}, \quad (i=1,\ldots,r). \tag{2.5}$$

and its isospectral deformation equations

$$\frac{\partial}{\partial t_n^{(i)}} = B_n^{(i)}\Psi \qquad (i=1,\ldots,r;\ n=1,2,\ldots)$$

where $B_n^{(i)}$ are "differential operator" part of $C_i L^n$; e.g.

$$B_1^{(i)} = E_{ii}\frac{d}{dx} + C_{i1}, \quad B_2^{(i)} = E_{ii}\left(\frac{d}{dx}\right)^2 + C_{i1}\frac{d}{dx} + (2E_{ii}U_2 + C_{i2}), \ldots .$$

Hence the isospectral deformation of $L, C_1, \ldots, C_r$ is given by Lax-type equations

$$\frac{\partial L}{\partial t_n^{(i)}} = [B_n^{(i)}, L], \qquad \frac{\partial C_j}{\partial t_n^{(i)}} = [B_n^{(i)}, C_j], \tag{2.6}$$

$$(i,j=1,\ldots,r; \quad n=1,2,\ldots),$$

or equivalently, by Zakharov-Shabat type equations

$$\left[B_n^{(i)} - \frac{\partial}{\partial t_n^{(i)}}, \quad B_m^{(j)} - \frac{\partial}{\partial t_m^{(j)}}\right] = 0, \tag{2.7}$$

$$(i,j=1,\ldots,r; \quad n,m=1,2,\ldots).$$

Again, we can find a matrix-coefficiented micro-differential operator

$$W = 1 + W_1(x,t)\left(\frac{d}{dx}\right)^{-1} + W_2(x,t)\left(\frac{d}{dx}\right)^{-2} + \ldots \tag{2.8}$$

so that $L, C_1, \ldots, C_r$ are transformed to trivial ones, $\frac{d}{dx}, E_{11}, \ldots, E_{rr}$:

$$L = W \cdot \frac{d}{dx} \cdot W^{-1}, \qquad C_i = W \cdot E_{ii} \cdot W^{-1}, \qquad (i=1,\ldots,r), \tag{2.9}$$

and we get

$$\Psi(x,t) = W\Psi_0(x,t)$$
$$= (1+W_1(x,t)\lambda^{-1}+W_2(x,t)\lambda^{-2}+\ldots)\Psi_0(x,t) \tag{2.10}$$

where

$$\Psi_0(x,t) = \text{constant} \times \begin{pmatrix} e^{\lambda x+\eta(t^{(1)},\lambda)} & & \\ & \ddots & \\ & & e^{\lambda x+\eta(t^{(r)},\lambda)} \end{pmatrix}, \tag{2.11}$$

$$\text{with} \quad \eta(t^{(i)},\lambda) \underset{\text{def}}{=} \sum_{n=1}^{\infty} \lambda^n t_n^{(i)}. \tag{2.12}$$

§3 Grassmann manifolds of finite and infinite dimensions

Let $V = \mathbb{C}^{m+n}$ be a vector space of dimension $m+n$ over $\mathbb{C}$. (Everything in this note is defined over $\mathbb{C}$). The Grassmann manifold $GM(m,n)$ (or $GM(m,V)$) is by definition the parameter space for the set of m-dimensional subspaces in V. Since such a subspace is spanned by an m-frame $\xi = (\xi^{(1)},\ldots,\xi^{(m)})$ consisting m linearly independent vectors $\xi^{(1)},\ldots,\xi^{(m)} \in V$, we have

$$GM(m,n) = \{\text{m-frames in } V\}/GL(m)$$
$$= GL(m+n)/GL(m,n) \tag{3.1}$$

where we denote by $GL(m,n)$ the subgroup of $GL(m+n)$ consisting of elements of the form $g = \begin{pmatrix} g_1 & g_2 \\ 0 & g_4 \end{pmatrix}$ with $g_1 \in GL(m)$, $g_4 \in GL(n)$. We also set

$$\widetilde{GM}(m,n) = (\{m\text{-frames}\} \times GL(1))/GL(m)$$

$$(= \{m\text{-frames}\}/SL(m), \quad \text{if} \quad m > 0) \tag{3.2}$$

$$\Lambda^m(V) = m\text{-th exterior product space of } V \tag{3.3}$$

and have the following situation

$$\begin{array}{ccc} \widetilde{GM}(m,n) & \hookrightarrow & \Lambda^m(V) - \{0\} \\ \Big\downarrow GL(1) & & \Big\downarrow GL(1) \\ GM(m,n) & \hookrightarrow & \begin{array}{l}\text{Projective space} \\ \text{of dimension } \binom{m+n}{m}-1\end{array} \end{array} \tag{3.4}$$

where the embedding of the upper line is defined by letting $\bar{\xi} \in \widetilde{GM}(m,n)$ represented by an m-frame $\xi = (\xi^{(1)},\ldots,\xi^{(m)})$ correspond to $\xi^{(1)}\wedge\ldots\wedge\xi^{(m)} \in \Lambda^m(V)-\{0\}$. (3.4) gives the standard way of embedding $GM(m,n)$ into a projective space. $\widetilde{GM}(m,n)$ will be called the standard line bundle over $GM(m,n)$. By denoting by $\xi_{\ell_1\ldots\ell_m}$ the minors of ξ, i.e. the determinants of $m \times m$ matrices consisting of ℓ_1-th,..., ℓ_m-th rows of the $(m+n)$ $m \times$ matrix ξ, we have

$$\xi^{(1)}\wedge\ldots\wedge\xi^{(m)} = \sum_{1 \leq \ell_1 < \ldots < \ell_m \leq m+n} \xi_{\ell_1\ldots\ell_m} e_{\ell_1}\wedge\ldots\wedge e_{\ell_m} \tag{3.5}$$

where $e_1,\ldots,e_{m+n}$ denote the unit column vectors $\in V$. $\xi_{\ell_1,\ldots,\ell_m}$ are the Plücker coordinates of $\bar{\xi}$, and satisfy the following Plücker's identities:

$$\sum_{i=1}^{m+1} (-)^i \xi_{\ell_1 \ldots \ell_{m-1} k_i} \xi_{k_1 \ldots \hat{k}_i \ldots k_{m+1}} = 0. \tag{3.6}$$

In (3.4), GM(m,n) coincides with the intersection of the quadrics defined by (3.6) in the projective space. We remind that the derivation of (3.6) relies on the Clifford algebra structure of $\Lambda(V^*)\cdot\Lambda(V)$.

We denote by V^* the dual vector space of V. The canonical inner product $<\ ,\ > : V^* \times V \to \mathbb{C}$ is extended to $\Lambda^m(V^*) \times \Lambda^m(V) \to \mathbb{C}$ and it takes the form

$$<\bar{\xi}^*, \bar{\xi}> = \det{}^t\xi^*\cdot\xi = \sum_{\ell_1<\ldots<\ell_m} \xi^*_{\ell_1\ldots\ell_m}\xi_{\ell_1\ldots\ell_m} \tag{3.6}$$

on $\widetilde{GM}(m,V^*) \times \widetilde{GM}(m,V)$. For $m \leq m'$, $n \leq n'$ we regard $V = \mathbb{C}^{m+n}$ as a subspace of $V' = \mathbb{C}^{m'+n'}$ according to the scheme $V' = \mathbb{C}^{m'-m} \oplus V \oplus \mathbb{C}^{n'-n}$, and let an m-dimensional subspace V_1 of V correspond to the m'-dimensional subspace $\mathbb{C}^{m'-m} \oplus V_1 \oplus 0$ of V'. This process induces the embeddings

$$\begin{array}{ccc} \widetilde{GM}(m,n) & \hookrightarrow & \widetilde{GM}(m'n') \\ \cup\!\!\!| & & \cup\!\!\!| \\ \bar{\xi} & \longmapsto & \bar{\xi}' \end{array} \quad \text{and} \quad \begin{array}{ccc} GL(m+n) & \hookrightarrow & GL(m'+n') \\ \cup\!\!\!| & & \cup\!\!\!| \\ g & \longmapsto & g' \end{array} \tag{3.8}$$

where

$$\xi' = \begin{array}{c} \\ m'-m \\ m+n \\ n'-n \end{array} \begin{array}{cc} \overbrace{\quad}^{m'-m} & \overbrace{\quad}^{m} \\ \multicolumn{2}{c}{\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \xi \\ \hline 0 & 0 \end{array}\right)} \end{array} \quad \text{and} \quad g' = \begin{array}{ccc} \overbrace{\quad}^{m'-m} & \overbrace{\quad}^{m+n} & \overbrace{\quad}^{n'-n} \\ \multicolumn{3}{c}{\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & g & 0 \\ \hline 0 & 0 & 1 \end{array}\right)} \end{array}. \tag{3.9}$$

If, further, $\bar{\xi}^* \in \widetilde{GM}(m,V^*)$ goes to $\bar{\xi}^{*\prime} \in \widetilde{GM}(m',V'^*)$ then

$$<\bar{\xi}^*, \bar{\xi}> = <\bar{\xi}^{*\prime}, \bar{\xi}'>, \tag{3.10}$$

namely, the inner product is preserved by this embedding process.

The infinite dimensional Grassmann manifold (GM) and its standard line bundle ($\widetilde{GM}$) which we need to parametrize the solutions of KP hierarchy are obtained as the topological closure of the inductive limit of $GM(m,n)$ and $\widetilde{GM}(m,n)$ as m and n tend to ∞. Explicitly, our GM is defined by

$$GM = \{\mathbb{N}^c\text{-frames}\}/GL(\mathbb{N}^c), \tag{3.11}$$

where by an $\mathbb{N}^c$-frame we mean an infinite-sized matrix

$$\xi = (\xi_{\mu\nu})_{\mu\in\mathbb{Z},\nu\in\mathbb{N}^c} \tag{3.12}$$

whose rows and columns are labeled by integers $\mathbb{Z}$ and strictly negative integers $\mathbb{N}^c = \mathbb{Z} - \mathbb{N} = \{-1, -2, \ldots\}$, respectively,
det
satisfying the condition that (i) $\exists m \in \mathbb{N}$ such that $\xi_{\mu\nu} = \delta_{\mu\nu}$ for $\mu < -m$, and that (ii) m column vectors for $\nu = -m, -m+1, \ldots, -1$ are linearly independent, while $GL(\mathbb{N}^c)$

consists of

$$h = (h_{\mu\nu})_{\mu,\nu \in \mathbb{N}^c} \tag{3.13}$$

satisfying the similar condition as above.

§4. Construction of solutions to KP hierarchy

Let Λ be the shift operator defined by $\Lambda = (\delta_{\mu,\nu-1})_{\mu,\nu\in\mathbb{Z}}$. We define τ-function for KP hierachy by

$$\tau(t;\bar{\xi}) = \tau(t_1,t_2,\ldots;\bar{\xi}) \underset{\text{def}}{=} \langle\bar{\xi}_0, e^{\eta(t,\Lambda)}\bar{\xi}\rangle \tag{4.1}$$

where $\eta(t,\Lambda) \underset{\text{def}}{=} \sum_{n=1}^{\infty} t_n\Lambda^n$, and $\xi_0 \underset{\text{def}}{=} (\delta_{\mu\nu})_{\mu\in Z,\nu\in\mathbb{N}^c}$. It is shown that (4.1) well defines $\tau(t;\bar{\xi})$ under a very mild condition on the growth order of $(t_n)_{n=1,2,\ldots}$ and $(\xi_{\mu\nu})_{\mu\in Z,\nu\in\mathbb{N}^c}$.

From (4.1) we get the following expansion of τ in terms of character polynomials $\chi_Y(t)$ (cf. Appendix):

$$\tau(t;\bar{\xi}) = \sum_Y \xi_Y\chi_Y(t) \tag{4.2}$$

$$= \xi_\phi + \xi_{\square}\chi_{\square}(t) + \xi_{\square\square}\chi_{\square\square}(t) + \xi_{\boxminus}\chi_{\boxminus}(t) + \ldots .$$

What distingishes (4.2) from the expansion (A3) of an arbitrary function $f(t)$ is the fact that coefficients ξ_Y in (4.2) are Plücker coordinates of $\bar{\xi}\in\widetilde{GM}$, and as such they satisfy the Plücker identities, e.g.

$$\xi_{\phi}\xi_{\ydiagram{2,2}} - \xi_{\ydiagram{1,1}}\xi_{\ydiagram{2,1}} + \xi_{\ydiagram{2}}\,\xi_{\ydiagram{1,1}} = 0, \text{ etc.} \tag{4.3}$$

To any $\bar{\xi} \in \widetilde{GM}$ we associate $\overline{\xi[\lambda]} \in \widetilde{GM}$ defined by

$$(\xi[\lambda])_{\mu\nu} = \begin{cases} \xi_{\mu+1,\nu+1} & \text{if } \nu < -1 \\ \\ \lambda^{\mu+1} & \text{if } \nu = -1, \end{cases} \tag{4.4}$$

and define $\Psi(t;\lambda;\bar{\xi})$ by

$$\Psi(t;\lambda;\bar{\xi}) = \frac{\tau(t;\overline{\xi[\lambda]})}{\tau(t;\bar{\xi})} . \tag{4.5}$$

Then we have

<u>Theorem.</u>

For any $\bar{\xi} \in \widetilde{GM}$ (4.5) solves KP hierarchy (1.2), (1.3), and vice versa.

Namely, $\bar{\xi} \in \widetilde{GM}$ completely parametrizes the set of solutions to KP hierarchy.

$u_\nu(t)$ are given in terms of τ. For example:

$$u_2(t;\xi) = (\log\tau)_{t_1 t_1}, \quad u_3(t,\xi) = \frac{-1}{2}(\log\tau)_{t_1 t_2} + \frac{1}{2}(\log\tau)_{t_1 t_1 t_1} \cdots .$$

Returning to the τ-functions, we have

$$\tau(t+a,\bar{\xi}) = \tau(t, e^{\eta(a,\Lambda)}\bar{\xi}) \tag{4.6}$$

$$e^{\sum_{n=1}^{\infty} nb_n t_n}\tau(t,\xi) = \tau(t,e^{\eta(b,{}^t\Lambda)}\xi) \qquad (4.7)$$

which are very special cases of the general transformation formula of the form

$$T_g\tau(t,\xi) = \tau(t,g\xi) \qquad (4.8)$$

where T_g is a linear operator acting on the functions in t. Let K^+ and K^- denote the abelian subgroups of $GL(\infty)$ consisting of elements of the form $e^{\eta(a,\Lambda)}$ and $e^{\eta(b,{}^t\Lambda)}$, respectively, and set $K = K^+\cdot K^-$. (4.6) shows in particular that $\tau(t,\xi) = \tau(0,e^{\eta(t,\Lambda)}\xi)$ and hence, that the evolution of τ (and hence, of u_ν) in variables $t_1, t_2,\ldots$ is interpreted as the motion of $\xi \in \widetilde{GM}$ caused by the action of $e^{\eta(t,\Lambda)} \in K^+ = K/K^-$. On the other hand, (4.7) implies that the action of $e^{\eta(b,{}^t\Lambda)} \in K^-$ on ξ does not change $\Psi(t;\lambda;\xi)$ and $u_\nu(t;\xi)$. (This in particular means that what is in 1-1 corresponce with the totality of solutions of KP hierarchy is the quotient space $K^-\backslash GM$, rather than GM or $\widetilde{GM}$ itself.)

§5. Specializations

If $\xi \in \widetilde{GM}$ is generic, its orbit $K\xi$ is dense in $\widetilde{GM}$. If, contrarily, ξ lies in a submanifold of $\widetilde{GM}$ which is stable under K, ξ represents a solution of special type.

Let $f(\lambda) \in \mathbb{C}((\lambda^{-1}))$ (= the field of quotients in the formal power series ring $\mathbb{C}[[\lambda^{-1}]]$). Then, regarding $f(\Lambda)$

as an element of the Lie algebra of K, we have

$$f(\Lambda) \text{ fixes } \bar{\xi} \iff f(L) \text{ is a differential operator.} \tag{5.1}$$

Let A be a subalgebra of $\mathbb{C}((\lambda^{-1}))$ such that

$$A \cap \mathbb{C}[[\lambda^{-1}]] = \mathbb{C}, \tag{5.2}$$

and define

$$\widetilde{GM}^A \underset{\mathrm{def}}{=} \{\bar{\xi} \in \widetilde{GM} \mid f(\Lambda) \text{ fixes } \bar{\xi} \text{ for any } f(\lambda) \in A\}. \tag{5.3}$$

Then we see that $\widetilde{GM}^A$ is the parameter space for solutions of the specialized KP hierarchy satisfying the additional condition:

$$f(L) \text{ is a differential operator for } {}^{\forall} f(\lambda) \in A. \tag{5.4}$$

For example, if A contains an element of the form $\lambda^n + a_1\lambda^{n-1} + \dots$ for every sufficiently large n, then $\widetilde{GM}^A$ represents quasi-periodic, soliton or rational solutions attached to the algebraic curve SpecA, while $\widetilde{GM}^{\mathbb{C}[\lambda^2]}$ and $\widetilde{GM}^{\mathbb{C}[\lambda^3]}$ parametrize solutions to KdV hierarchy and Bonssinesq hierarchy, respectively.

Appendix. Character polynomials of GL

Irreducible tensor representation of $GL(n)$ is classified by Young diagrams. Let $\chi_Y(g)$ be the irreducible character of $g \in GL(n)$ corresponding to a Young diagram Y. Since $\chi_Y(g)$ is a symmetric polynomial of eigenvalues $\varepsilon_1, \ldots, \varepsilon_n$ of g, it is expressed as a polynomial of $t_1, t_2, \ldots$ defined by

$$t_\nu = \frac{1}{\nu}(\varepsilon_1^\nu + \ldots + \varepsilon_n^\nu) = \frac{1}{\nu}\text{trace } g^\nu. \tag{A1}$$

The result is well known to be

$$\chi_Y(g) = \chi_Y(t) = \sum_{\nu_1+2\nu_2+\ldots=N} \pi_Y(1^{\nu_1}2^{\nu_2}\ldots)\frac{t_1^{\nu_1}t_2^{\nu_2}\ldots}{\nu_1!\nu_2!\ldots}, \tag{A2}$$

where N = size of Y (= rank of tensor), and $\pi_Y(1^{\nu_1}2^{\nu_2}\ldots)$ denotes the irreducible character of the symmetric permutation group of N letters corresponding to the Young diagram Y and to the conjugacy class consisting ν_1 cycles of size 1 ν_2 cycles of size 2, etc. For example we have

$$\chi_\phi(t) = 1, \quad \chi_{\square}(t) = t_1, \quad \chi_{\square\square}(t) = \frac{t_1^2}{2}+t_2, \quad \chi_{\substack{\square\\\square}}(t) = \frac{t_1^2}{2}-t_2,$$

$$\chi_{\square\square\square}(t) = \frac{t_1^3}{6}+t_1t_2+t_3, \quad \chi_{\substack{\square\square\\\square}}(t)=\frac{t_1^3}{3}-t_3, \quad \chi_{\substack{\square\\\square\\\square}}(t)=\frac{t_1^3}{6}-t_1t_2+t_3, \ldots .$$

Note that these "character polynomials" $\chi_Y(t)$ are independent of n as long as $n \geq N$ so that the variables $t_1, \ldots, t_N$ are independent.

$\{\chi_Y(t)\}$ constitute a linear base for the polynomial ring $\mathbb{C}[t_1,t_2,\ldots]$. Namely, any polynomial $f(t)\in\mathbb{C}[t_1,t_2,\ldots]$ is uniquely expressed as a finite linear sum of $\chi_Y(t)$'s. Further, any formal power series $f(t)\in\mathbb{C}[[t_1,t_2,\ldots]]$ is uniquely expressed as an infinite formal sum of $\chi_Y(t)$'s:

$$f(t) = \sum_Y f_Y \cdot \chi_Y(t), \quad f_Y \in \mathbb{C}, \tag{A3}$$

and vice versa.

It is known that Young diagrams are in 1-1 correspondence with strictly monotone maps $\sigma : \mathbb{N}^c \to \mathbb{Z}$ such that $\sigma(\nu) = \nu$ for almost all $\nu \in \mathbb{N}^c$. On the other hand, such σ is characterized by the strictly increasing series of natural numbers $(\ell_0, \ell_1, \ldots, \ell_{n-1})$ defined by $\ell_\nu = \sigma(-n+\nu)$, $\nu = 0, 1, \ldots, n-1$, where we assume that n is so chosen that $\sigma(\nu) = \nu$ for $\nu < -n$.